Blow Up of Solutions for a System of Nonlinear Viscoelastic Equations with Damping Terms in $\mathbb{R}^n$

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We consider a coupled system of nonlinear viscoelastic equations with linear damping and source terms. Under suitable conditions of the initial data and the relaxation functions, we prove a finite-time blow-up result with vanishing initial energy by using the modified energy method and a crucial lemma on differential inequality.

Key words: Blow Up; Coupled System; Nonlinear Viscoelastic Equation; Damping Term.
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1. Introduction

We consider the Cauchy problem for the following coupled system of nonlinear viscoelastic equations with linear damping and source terms:

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s + u_t = f_1(u,v), \\ &\qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ &v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)\,\mathrm{d}s + v_t = f_2(u,v), \\ &\qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \\ &v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where g, h, u_0, u_1, v_0, v_1 are functions to be specified later. This type of problems arises naturally in the theory of viscoelasticity and describes the interaction of two scalar fields (see [1,2]). The integral terms express the fact that the stress at any instant depends not only on the present value but on the entire past history of strains the material has undergone.

The motivation of our work is due to the initial boundary problem of the scalar equation

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + u_t|u_t|^{m-2} = u|u|^{p-2}, \\ &\qquad (x,t) \in \Omega \times (0,\infty), \\ &u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned} \tag{2}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function. In [3], Messaoudi showed, under suitable conditions of g, that solutions with negative initial energy blow up in finite time, if $p > m$, and continue to exist, if $m \geq p$. This result has been later pushed by the same author [4] to certain solutions with positive initial energy. We would like to mention that [1] was one of the first papers considering the viscoelastic equation

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau - \Delta u_t = 0$$

with Dirichlet boundary conditions, which showed uniform decay rates of the energy subject to a strong dissipative term. And [5] is the most recent paper on this subject taking into account the contrast of frictional versus viscoelastic effects, in which the authors established general decay rates for the viscoelastic wave equation strongly weakening the usual assumptions on the relaxation function. In the absence of the viscoelastic term ($g = 0$), (2) has also been extensively studied and many results concerning the global existence and nonexistence have been proved (see [6 – 13] and the references therein).

In all above treatments the underlying domain is assumed to be bounded. The boundedness of the do-

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main is essential because of the usage of the boundedness of the injection $L^p(\Omega) \subset L^q(\Omega)$, when $1 \le q \le p$ (see [14, 15]). For problem (2) in $\mathbb{R}^n$, we also mention the work of Levine, Park and Serrin [16], Messaoudi [17], and Zhou [18]. Recently, Kafini and Messaoudi [19] studied the coupled system (1) but without damping terms. By defining the functional

$$F(t) = \frac{1}{2}\int_{\mathbb{R}^n}[|u(x,t)|^2 + |v(x,t)|^2]\,\mathrm{d}x + \frac{1}{2}\beta(t+t_0)^2 \quad (3)$$

and using the classical concavity method, they proved that the solution blows up in finite time if the initial energy is negative. More recently, the same authors [20] considered the following Cauchy problem with a damping term:

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s + u_t = u|u|^{p-2}, \\ &\quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \end{aligned} \quad (4)$$

Applying suitable conditions to the initial data and the condition

$$\int_0^{+\infty} g(s)\,\mathrm{d}s < \frac{p-2}{p-3/2} \quad (5)$$

to the relaxation function, they proved a blow-up result with vanishing initial energy.

Motivated by the ideas of [3, 19, 20], we intend to extend the result of [19, 20] to our problem (1). We shall prove a finite-time blow-up result for problem (1) with vanishing initial energy. We note that the method used in [19] cannot be applied to our problem directly since the damping terms are contained. To achieve our goal we will use the functional (14) below [instead of (3)] and modify the method of [20] [see (19) – (21) below] so that the blow-up result for a single equation is extended to the coupled system (1). Moreover, our assumption for the relaxation functions [see (10) below] is slightly weaker than that of [20]. The lack of the injection $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ shall be compensated by the usage of the compact support method. The results obtained in the present paper might find some potential applications in the theory of nonlinear viscoelasticity.

The manuscript is organized as follows. In Section 2 we make some assumptions for the relaxation functions g, h and the coupled terms. The local existence of solutions and a crucial lemma are also stated. Our main result is given and proved in Section 3.

2. Preliminaries

In this section we present some material needed in the proof of our main result. First, we make the following assumptions:

(G1) $g,h : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing differentiable functions satisfying

$$1 - \int_0^\infty g(s)\,\mathrm{d}s = l > 0, \quad t \ge 0,$$
$$1 - \int_0^\infty h(s)\,\mathrm{d}s = k > 0, \quad t \ge 0.$$

(G2) There exists a function $I(u,v) \ge 0$ such that

$$\frac{\partial I}{\partial u} = f_1(u,v), \quad \frac{\partial I}{\partial v} = f_2(u,v).$$

(G3) There exist positive constants B and ρ such that $2 < \rho < 2 + 2/n$ and

$$\int_{\mathbb{R}^n}[uf_1(u,v) + vf_2(u,v)]\,\mathrm{d}x \ge \rho\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x \ge B\int_{\mathbb{R}^n}(|u|^\rho + |v|^\rho)\,\mathrm{d}x.$$

(G4) There exists a constant $d > 0$ such that

$$|f_1(\xi,\varsigma)| \le d(|\xi|^{\gamma_1} + |\varsigma|^{\gamma_2}), \quad \forall(\xi,\varsigma) \in \mathbb{R}^2,$$
$$|f_2(\xi,\varsigma)| \le d(|\xi|^{\gamma_3} + |\varsigma|^{\gamma_4}), \quad \forall(\xi,\varsigma) \in \mathbb{R}^2,$$

where

$$\gamma_i \ge 1, \quad (n-2)\gamma_i \le n, \quad i = 1,2,3,4.$$

Remark. (G1) is necessary to guarantee the hyperbolicity of system (1). Condition (G4) is necessary for the existence of a local solution to (1). As an example of functions satisfying (G2) – (G4), we have

$$I(u,v) = \frac{1}{\rho}(|uv|^\rho + |u|^\rho + |v|^\rho),$$
$$(n-2)\rho \le 2(n-1).$$

We introduce the “modified” energy functional, as in [19],

$$\begin{aligned} E(t) := &\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 \\ &+ \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \\ &+ \frac{1}{2}\left(1 - \int_0^t h(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2 + \frac{1}{2}(g\circ\nabla u) \\ &+ \frac{1}{2}(h\circ\nabla v) - \int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x, \end{aligned} \quad (6)$$

where

$$(g\circ\nabla u)(t)=\int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^2\,\mathrm{d}\tau,$$
$$(h\circ\nabla v)(t)=\int_0^t h(t-\tau)\|\nabla v(t)-\nabla v(\tau)\|_2^2\,\mathrm{d}\tau. \tag{7}$$

We now state, without a proof, a local existence result, which can be established by combining the arguments of [1, 7, 14].

Theorem 1. Assume that (G1) and (G4) hold. Then for the initial data $(u_0,v_0)\in[H^1(\mathbb{R}^n)]^2$, $(u_1,v_1)\in[L^2(\mathbb{R}^n)]^2$, with compact support, problem (1) has a unique local solution

$$(u,v)\in[C([0,T);\,H^1(\mathbb{R}^n))]^2,$$
$$(u_t,v_t)\in[C([0,T);\,L^2(\mathbb{R}^n)\cap L^2([0,T)\times\mathbb{R}^n)]^2$$

for T small enough.

In order to prove our main result, we need the following crucial lemma on differential inequality.

Lemma (see [21, Proposition 3.1]). Suppose that $G(t)$ is a twice continuously differentiable function satisfying

$$G''(t)+G'(t)\ge C_0(t+L)^\beta G^{1+\alpha}(t),\quad t>0, \tag{8}$$

$$G(0)>0,\quad G'(0)\ge 0,$$

where C_0, $L>0$, $-1<\beta\le 0$, $\alpha>0$ are constants. Then $G(t)$ blows up in finite time. Moreover, the blow-up time can be estimated as

$$T_0=\left(\frac{2G^{-\alpha/2}(0)}{\delta\alpha}+L^{\beta+1}\right)^{1/(\beta+1)}-L, \tag{9}$$

where $\delta>0$ is a small constant such that $\delta<G'(0)/[L^\beta G^{1+\alpha/2}(0)]$.

3. Blow Up of Solution with Vanishing Initial Energy

In this section we state and prove our main result.

Theorem 2. Assume that (G1)–(G4) hold and that

$$\max\left\{\int_0^{+\infty}g(s)\,\mathrm{d}s,\int_0^{+\infty}h(s)\,\mathrm{d}s\right\}<\frac{\rho-2}{\rho-2+1/\rho}. \tag{10}$$

Then for the initial data $(u_0,v_0),(u_1,v_1)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, with compact support, satisfying

$$E(0)=\frac12\|u_1\|_2^2+\frac12\|\nabla u_0\|_2^2+\frac12\|v_1\|_2^2+\frac12\|\nabla v_0\|_2^2-\int_{\mathbb{R}^n}I(u_0,v_0)\,\mathrm{d}x\le 0 \tag{11}$$

and

$$\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\,\mathrm{d}x\ge 0,$$

the corresponding solution of (1) blows up in finite time.

Proof. Multiplying the equations in (1) by u_t and v_t, respectively, and integrating over $\mathbb{R}^n$, we obtain [3]

$$E'(t)=-(\|u_t\|_2^2+\|v_t\|_2^2)+\frac12(g'\circ\nabla u)+\frac12(h'\circ\nabla v)-\frac12 g(s)\|\nabla u\|_2^2-\frac12 h(s)\|\nabla v\|_2^2\le 0. \tag{12}$$

Hence,

$$E(t)\le E(0)<0. \tag{13}$$

We then define

$$G(t)=\frac12\int_{\mathbb{R}^n}[|u(x,t)|^2+|v(x,t)|^2]\,\mathrm{d}x. \tag{14}$$

By differentiating G twice we get

$$G'(t)=\int_{\mathbb{R}^n}(u_tu+v_tv)\,\mathrm{d}x, \tag{15}$$

$$G''(t)=\int_{\mathbb{R}^n}(u_{tt}u+v_{tt}v)\,\mathrm{d}x+\int_{\mathbb{R}^n}(|u_t|^2+|v_t|^2)\,\mathrm{d}x. \tag{16}$$

To estimate the term $\int_{\mathbb{R}^n}(u_{tt}u+v_{tt}v)\,\mathrm{d}x$ in (16), we multiply the equations in (1) by u and v, respectively, and integrate them by parts over $\mathbb{R}^n$ to get

$$\begin{aligned}\int_{\mathbb{R}^n}(uu_{tt}+vv_{tt})\,\mathrm{d}x=\\ &-\int_{\mathbb{R}^n}(|\nabla u|^2+|\nabla v|^2)\,\mathrm{d}x\\ &+\int_{\mathbb{R}^n}[uf_1(u,v)+vf_2(u,v)]\mathrm{d}x\\ &+\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(x,t).\nabla u(x,s)\,\mathrm{d}x\,\mathrm{d}s\\ &+\int_0^t h(t-s)\int_{\mathbb{R}^n}\nabla v(x,t).\nabla v(x,s)\,\mathrm{d}x\,\mathrm{d}s\\ &-\int_{\mathbb{R}^n}(u_tu+v_tv)\,\mathrm{d}x.\end{aligned}$$

But

$$\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(x,t).\nabla u(x,s)\,\mathrm{d}x\,\mathrm{d}s = \int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(x,t).[\nabla u(x,s)-\nabla u(x,t)]\,\mathrm{d}x\,\mathrm{d}s + \left(\int_0^t g(s)\,\mathrm{d}s\right)\int_{\mathbb{R}^n}|\nabla u(x,t)|^2\,\mathrm{d}x.$$

Using Young's inequality and (G3) we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n}(uu_{tt}+vv_{tt})\,\mathrm{d}x \geq{}& \left[-1-\delta+\int_0^t g(s)\,\mathrm{d}s\right]\|\nabla u\|_2^2 \\ &+\rho\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x-\frac{1}{4\delta}\left(\int_0^t g(s)\,\mathrm{d}s\right)(g\circ\nabla u) \\ &+\left[-1-\delta+\int_0^t h(s)\,\mathrm{d}s\right]\|\nabla v\|_2^2 \\ &-\frac{1}{4\delta}\left(\int_0^t h(s)\,\mathrm{d}s\right)(h\circ\nabla v)-\int_{\mathbb{R}^n}(u_tu+v_tv)\,\mathrm{d}x\end{aligned}\tag{17}$$

for all $\delta>0$. By combining (15)–(17), we get

$$\begin{aligned}G''(t)+G'(t)\geq{}& \left[-1-\delta+\int_0^t g(s)\,\mathrm{d}s\right]\|\nabla u\|_2^2 \\ &-\frac{1}{4\delta}\left(\int_0^t g(s)\,\mathrm{d}s\right)(g\circ\nabla u) \\ &+\left[-1-\delta+\int_0^t h(s)\,\mathrm{d}s\right]\|\nabla v\|_2^2 \\ &-\frac{1}{4\delta}\left(\int_0^t h(s)\,\mathrm{d}s\right)(h\circ\nabla v) \\ &+\rho\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x+\int_{\mathbb{R}^n}(|u_t|^2+|v_t|^2)\,\mathrm{d}x.\end{aligned}\tag{18}$$

By using (6), that is

$$\begin{aligned}&\left(-1+\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 +\left(-1+\int_0^t h(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2 = \\ &-2E(t)+(\|u_t\|_2^2+\|v_t\|_2^2)+(g\circ\nabla u) +(h\circ\nabla v)-2\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x,\end{aligned}$$

(18) becomes

$$\begin{aligned}G''(t)+G'(t)\geq{}& -2E(t)-\delta\|\nabla u\|_2^2 \\ &+\left[1-\frac{1}{4\delta}\left(\int_0^t g(s)\,\mathrm{d}s\right)\right](g\circ\nabla u) \\ &-\delta\|\nabla v\|_2^2+\left[1-\frac{1}{4\delta}\left(\int_0^t h(s)\,\mathrm{d}s\right)\right](h\circ\nabla v) \\ &+(1-\gamma)(\rho-2)\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x \\ &+\gamma(\rho-2)\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x+2\int_{\mathbb{R}^n}(|u_t|^2+|v_t|^2)\,\mathrm{d}x\end{aligned}\tag{19}$$

for all $0<\gamma<1$. Now, we exploit (6) to substitute $(1-\gamma)(\rho-2)\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x$, thus (19) takes the form

$$\begin{aligned}&G''(t)+G'(t)\geq \\ &\left[\frac{(1-\gamma)(\rho-2)}{2}\left(1-\int_0^t g(s)\,\mathrm{d}s\right)-\delta\right]\|\nabla u\|_2^2 \\ &+\left[\frac{(1-\gamma)(\rho-2)}{2}\left(1-\int_0^t h(s)\,\mathrm{d}s\right)-\delta\right]\|\nabla v\|_2^2 \\ &+\left[1+\frac{(1-\gamma)(\rho-2)}{2}-\frac{1}{4\delta}\left(\int_0^t g(s)\,\mathrm{d}s\right)\right](g\circ\nabla u) \\ &-(2+(1-\gamma)(\rho-2))E(t) \\ &+\left[1+\frac{(1-\gamma)(\rho-2)}{2}-\frac{1}{4\delta}\left(\int_0^t h(s)\,\mathrm{d}s\right)\right](h\circ\nabla v) \\ &+\gamma(\rho-2)\int_{\mathbb{R}^n} I(u,v)\mathrm{d}x \\ &+\left(2+\frac{(1-\gamma)(\rho-2)}{2}\right)\int_{\mathbb{R}^n}(|u_t|^2+|v_t|^2)\,\mathrm{d}x.\end{aligned}\tag{20}$$

Next, we choose $\delta>0$ so that

$$\frac{(1-\gamma)(\rho-2)}{2}\left(1-\int_0^\infty g(s)\,\mathrm{d}s\right)-\delta\geq 0,$$
$$1+\frac{(1-\gamma)(\rho-2)}{2}-\frac{1}{4\delta}\left(\int_0^\infty g(s)\,\mathrm{d}s\right)\geq 0,$$

and

$$\frac{(1-\gamma)(\rho-2)}{2}\left(1-\int_0^\infty h(s)\,\mathrm{d}s\right)-\delta\geq 0,$$
$$1+\frac{(1-\gamma)(\rho-2)}{2}-\frac{1}{4\delta}\left(\int_0^\infty h(s)\,\mathrm{d}s\right)\geq 0.$$

This is, of course, possible by (10). We then conclude, from (20), that

$$G''(t)+G'(t)\geq\gamma(\rho-2)\int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x,\quad \forall t\geq 0.\tag{21}$$

Now, we use the finite speed of propagation for system (1) and Hölder's inequality to obtain

$$\int_{\mathbb{R}^n} |u|^2 \,\mathrm{d}x \le \left(\int_{\mathbb{R}^n} |u|^\rho \,\mathrm{d}x\right)^{\frac{2}{\rho}} \left(\int_{B(t+L)} 1 \,\mathrm{d}x\right)^{\frac{\rho-2}{\rho}} \tag{22}$$

and

$$\int_{\mathbb{R}^n} |v|^2 \,\mathrm{d}x \le \left(\int_{\mathbb{R}^n} |v|^\rho \,\mathrm{d}x\right)^{\frac{2}{\rho}} \left(\int_{B(t+L)} 1 \,\mathrm{d}x\right)^{\frac{\rho-2}{\rho}}, \tag{23}$$

where the constant $L > 0$ is such that

$$\operatorname{supp}\{u_0(x), u_1(x), v_0(x), v_1(x)\} \subset \{|x| \le L\}$$

and $B(t+L)$ is the ball, with radius $t+L$, centered at the origin. If we call W_n the volume of the unit sphere in $\mathbb{R}^n$, then

$$\begin{aligned}
&\int_{\mathbb{R}^n} (|u|^\rho + |v|^\rho) \,\mathrm{d}x \ge \\
&\quad \left(\int_{\mathbb{R}^n} |u|^2 \,\mathrm{d}x\right)^{\frac{\rho}{2}} (W_n(t+L)^n)^{-\frac{\rho-2}{2}} \\
&\quad + \left(\int_{\mathbb{R}^n} |v|^2 \,\mathrm{d}x\right)^{\frac{\rho}{2}} (W_n(t+L)^n)^{-\frac{\rho-2}{2}} \\
&= W_n^{-\frac{\rho-2}{2}} (t+L)^{-\frac{n(\rho-2)}{2}} \\
&\quad \cdot \left[\left(\int_{\mathbb{R}^n} |u|^2 \,\mathrm{d}x\right)^{\frac{\rho}{2}} + \left(\int_{\mathbb{R}^n} |v|^2 \,\mathrm{d}x\right)^{\frac{\rho}{2}}\right] \\
&\ge 2^{-\frac{\rho-2}{2}} W_n^{-\frac{\rho-2}{2}} (t+L)^{-\frac{n(\rho-2)}{2}} \\
&\quad \cdot \left(\int_{\mathbb{R}^n} |u|^2 \mathrm{d}x + \int_{\mathbb{R}^n} |v|^2 \,\mathrm{d}x\right)^{\frac{\rho}{2}} \\
&= 2 W_n^{-\frac{\rho-2}{2}} (t+L)^{-\frac{n(\rho-2)}{2}} G^{\frac{\rho}{2}}(t)
\end{aligned} \tag{24}$$

by using the inequality

$$(a+b)^p \le 2^{p-1}(a^p + b^p), \quad \text{for } a, b > 0, \quad p > 1.$$

Consequently, by (G3), we have

$$\begin{aligned}
&G''(t) + G'(t) \ge \\
&2\gamma \frac{\rho-2}{\rho} B W_n^{-\frac{\rho-2}{2}} (t+L)^{-\frac{n(\rho-2)}{2}} G^{\frac{\rho}{2}}(t).
\end{aligned} \tag{25}$$

It is easy to verify that the requirements of Lemma are satisfied if

$$\begin{aligned}
&C_0 = 2\gamma \frac{\rho-2}{\rho} B W_n^{-\frac{\rho-2}{2}} > 0, \\
&-1 < \beta = -\frac{n(\rho-2)}{2} < 0, \\
&\alpha = \frac{\rho-2}{2} > 0.
\end{aligned}$$

Therefore $G(t)$ blows up in finite time.

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